Int. J. Heat Mass Transfer. Vol. 9, pp. 433–440. Pergamon Press 1966. Printed in Great Britain

LAMINAR HEAT TRANSFER IN TUBE WITH NONLINEAR RADIANT HEAT-FLUX BOUNDARY CONDITION†

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(*Received* 29 *March* 1965 *and in revised form* 25 *August* 1965)

Abstract—The problem of radiant cooling of a fluid in laminar flow through a tube was described by a nonlinear integral equation, and an approximate solution obtained in terms of the Liouville–Neumann series. Results were also obtained by an exact iterative numerical solution. Local Nusselt numbers are presented as a function of dimensionless distance, $x = Z/Re\,Pr\,R$, and a dimensionless parameter, $\alpha = \epsilon\sigma T_0^3/k$. An empirical equation,

$$Nu(\alpha, x) = (0.928 - 0.023 \ln \alpha) Nu_q(x)$$

where Nu_q = Nusselt number for the constant heat flux case, was found to give results accurate to within ± 2 per cent for the ranges of variables of interest.

NOMENCLATURE

- a, b, empirical constants in equation (17);
- C_n, eigenconstants for constant heat-flux problem;
- k, thermal conductivity of fluid;
- Nu, Nusselt number;
- Pe, Péclet number;
- Pr, Prandtl number;
- Q, heat flux at $r = R$;
- R, radius of tube;
- Re, Reynolds number;
- R_n, eigenfunctions for constant heat flux problem, at $r = R$;
- r, radial coordinate;
- T, absolute temperature;
- U_m, velocity at tube center line;
- x, dimensionless distance, $= Z/(Re\,Pr\,R)$;
- Z, longitudinal coordinate.

Greek symbols

- α, dimensionless parameter $\epsilon\sigma T_0^3 R/k$;
- β_n, eigenvalues for constant heat-flux problem;
- ϵ, emissivity of tube wall;
- ρ, density of fluid;
- σ, Stefan–Boltzmann constant;
- τ, dimensionless temperature, $= T/T_0$.

Subscripts

- b, fluid bulk;
- 0, at entrance ($Z = 0$);
- q, constant heat flux case;
- w, at wall ($r = R$);
- *, arbitrary base temperature for equation (11).

† This work was performed under the auspices of the U.S. Atomic Energy Commission.

INTRODUCTION

THE PROBLEM of heat transfer with laminar flow in a tube has long been of interest to investigators and a number of solutions have been published dealing with a variety of boundary conditions. Those solutions which involve prescribed temperature boundary conditions were well reviewed by Tribus and Klein [1]. Solutions which involve prescribed heat-flux boundary conditions include the well-known work of Siegel, Sparrow, and Hallman [2]. A third class of problems, in which neither the wall temperature nor the wall heat flux is prescribed *per se* but instead the heat flux is prescribed as a function of the local wall temperature, has received much less attention. This is a more difficult problem in that the heat-transfer equation now involves the unknown variable (either temperature or heat flux) in an implicit rather than explicit form. Sideman *et al.* [3] and Stein [4] recently treated problems of this class. Of special interest is the problem of radiant *cooling* at the boundary wall, where the wall heat flux is proportional to the fourth power of the local wall temperature. This problem has become increasingly important with the advent of such high temperature systems as radiators in space power systems, high temperature liquid metal facilities, and high temperature gas flow systems. No analytical or numerical solution for this problem has been published to date.

ANALYSIS AND APPROXIMATE SOLUTION

Consider a fluid with constant physical properties in non-dissipative, laminar flow through a round tube of radius R. Heat transfer at the wall starts at axial position $Z = 0$ and is proportional to $T_w^4(Z)$ for $Z > 0$. It is assumed that axial conduction is negligible and that at $Z = 0$, the fluid enters with a fully established laminar velocity profile and with a uniform temperature, T_0. The governing heat-transfer equation for $Z \geqslant 0$ is then

$$\rho U_m\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial Z} = k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \quad (1)$$

and the initial and boundary conditions are

$$\text{For } Z \leqslant 0, \quad T = T_0$$

$$\text{For } Z \geqslant 0, \quad k\left[\frac{\partial T}{\partial r}\right]_{r=R} = Q(Z) = -\epsilon\sigma T_w^4(Z) \quad (3)$$

$$k\left[\frac{\partial T}{\partial r}\right]_{r=0} = 0. \quad (4)$$

Since equation (1) is linear, the solution can be obtained by superposition of the constant heat flux solution for infinitesimal changes in heat flux. The effect of a step increase in heat flux at Z' of magnitude $Q(Z')$ is known to be [2]:

$$T_{w,q}(Z, Z') - T_0 = \frac{R}{k}\left\{\tfrac{11}{24} + \frac{4(Z - Z')}{Re\,Pr\,R} + \sum_{n=1}^{\infty} C_n R_n \exp\left[-\beta_n^2\frac{(Z - Z')}{Re\,Pr\,R}\right]\right\}Q(Z'). \quad (5)$$

Applying Duhamel's superposition theorem [5] to obtain the solution for the case of a constantly changing heat flux,

$$T_w(Z) - T_0 = \int_0^Z Q(Z')\frac{\partial T_{w,q}(Z, Z')}{\partial Z}\,dZ'$$

$$= \frac{R}{k}\int_0^Z\left\{\frac{4}{Re\,Pr\,R} - \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp\left[-\beta_n^2\frac{(Z - Z')}{Re\,Pr\,R}\right]\right\}Q(Z')\,dZ'. \quad (6)$$

Combining with boundary condition (3) and writing in terms of dimensionless variables,

$$\tau_w(x) = 1 - \alpha\int_0^x \{4 - \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp[-\beta_n^2(x - x')]\}\,\tau_w^4(x')\,dx'. \quad (7)$$

The local bulk fluid temperature is obtained by a heat balance,

$$\tau_b(x) = 1 - 4\alpha\int_0^x \tau_w^4(x')\,dx'. \quad (8)$$

The local Nusselt number is then

$$Nu(x) = \frac{2RQ(x)}{k[\tau_w(x) - \tau_b(x)]\, T_0} \tag{9}$$

$$Nu(x) = \frac{2\tau_w^4(x)}{- \int_0^x \tau_w^4(x') \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp\left[-\beta_n^2 (x - x')\right] \mathrm{d}x'}. \tag{10}$$

Equations (7) and (10) are the formal solutions to the problem. Unfortunately, the unknown, $\tau_w(x)$, is not only implicit in equation (7) but is also nonlinear. No exact explicit solution has been found.

An approximate explicit solution can be obtained by linearization of the fourth-order term,

$$\tau_w^4 \cong \tau_*^4 + (\tau - \tau_*)\left[\frac{\mathrm{d}\tau_w^4}{\mathrm{d}\tau_w}\right]_{\tau_*}. \tag{11}$$

Substituting into equation (7),

$$\tau_w(x) = 1 + \alpha \int_0^x 3\tau_*^4 \left\{4 - \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp\left[-\beta_n^2 (x - x')\right]\right\} \mathrm{d}x'$$
$$- \alpha \int_0^x 4\tau_*^3 \left\{4 - \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp\left[-\beta_n^2 (x - x')\right]\right\} \tau_w(x')\, \mathrm{d}x'. \tag{12}$$

Equation (12) is an integral equation of Volterra's second kind and can be solved in terms of the Liouville–Neumann series [6]:

$$\tau_w(x) = f(x) + \lambda \int_0^x \sum_{i=0}^{\infty} \lambda^i K_{i+1}(x, x') f(x')\, \mathrm{d}x' \tag{13}$$

where,

$$\left.\begin{aligned}
f(x) &= 1 + 12\alpha\tau_*^4 x - 3\alpha\tau_*^4 \sum_{n=1}^{\infty} C_n R_n (1 - \exp\left[-\beta_n^2 x\right]) \\
\lambda &= -4\alpha\tau_*^3 \\
K_1(x, x') &= 4 - \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp\left[-\beta_n^2 (x - x')\right] \\
K_{i+1}(x, x') &= \int_0^x \int_0^x \ldots \int_0^x K_1(x, y_1)\, K_1(y_1, y_2) \ldots K_1(y_i, x')\, \mathrm{d}y_1\, \mathrm{d}y_2 \ldots \mathrm{d}y_i
\end{aligned}\right\} \tag{14}$$

Equation (13) is an approximate solution for the longitudinal temperature profile at the wall, which may be substituted into equation (10) to calculate local Nusselt numbers.

NUMERICAL SOLUTION

To obtain more exact results, equation (7) was also solved by an iterative numerical procedure with the aid of an IBM 7094 computer. In a small enough increment of x, $\tau_w(x)$ may be taken to be a constant at some mean value. Thus for the jth increment, let τ_{mj} = mean value of τ_w between x_j and x_{j-1}. Then

$$\int_{x_{j-1}}^{x_j} \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \exp\left[-\beta_n^2 (x - x')\right] \tau_w^4(x')\, \mathrm{d}x'$$
$$= \tau_{mj}^4 \sum_{n=1}^{\infty} C_n R_n \left\{\exp\left[-\beta_n^2 (x - x_j)\right]\right.$$
$$\left. - \exp\left[-\beta_n^2 (x - \quad {}_{1}\right]\right\} \tag{15}$$

and equation (7) can be written as

$$\left.\begin{aligned} \tau_w(x_j) = \\ 1 - \alpha \int_0^{x_{j-1}} \tau_w^4(x') \{4 - \sum_{n=1}^{\infty} C_n R_n \beta_n^2 \,. \\ \exp[-\beta_n^2(x - x')]\} \, dx' \\ - 4\alpha(x_j - x_{j-1}) \, \tau_{mj}^4 \\ + \alpha \tau_{mj}^4 \sum_{n=1}^{\infty} C_n R_n \\ \{1 - \exp[-\beta_n^2(x_j - x_{j-1})]\}. \end{aligned}\right\} \quad (16)$$

Starting with $\tau_w(x_j = 0) = 1$, equation (16) was used in an iterative procedure to calculate $\tau_w(x_1)$, $\tau_w(x_2)$, ... and so forth. Usually, $\tau_w(x_j)$ for any specific x_j converged in less than ten iterations. The size of the incremental x was varied until further subdivisions caused negligible change in the results. It was found that the greater the value of the parameter, α, the more subdivisions were required for convergence. In above equations, β_n, R_n, and C_n are eigenvalues, eigenfunctions (at $r = R$), and eigenconstants, respectively, for the constant heat flux problem. The numerical values for these quantities up to $n = 7$ are given in reference [2]. Reference [7] gives values up to $n = 20$ and presents asymptotic expressions for higher numbers. It was discovered that seven eigenvalues did not give sufficient accuracy at $x < 0{\cdot}002$. Fifty eigenvalues were used in the final computations.

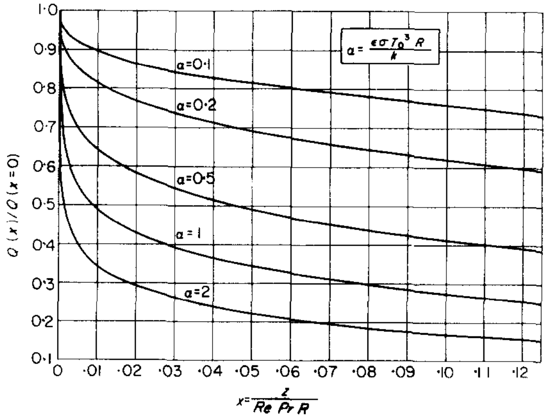

FIG. 1. Variation of local heat flux.

Results were obtained for α ranging from 0·1 to 50. Once the longitudinal wall temperature profile, $\tau_w(x)$, is determined, the variation in local wall heat flux can be calculated by equation (3). Some sample results are shown in Fig. 1. It is seen that heat flux decreases rapidly in the initial lengths, the decrease becoming more gradual and almost linear with x at greater distances. At high values of α, the heat flux is substantially reduced in very short distances. Thus for $\alpha = 1$, at $x = 0{\cdot}01$, the heat flux has decreased to 49 per cent of its initial value.

Table 1. Nu for laminar flow with radiant boundary condition

$\frac{Z}{Re\,Pr\,R}$	Local Nusselt numbers $\alpha = 0{\cdot}1$	$\alpha = 0{\cdot}2$	$a = 0{\cdot}5$	$a = 1$	$\alpha = 2$	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$	$\alpha = 50$
0·001	15·7	15·6	15·3	15·1	14·8	14·3	14·0	13·8	13·5
0·002	12·4	12·3	12·1	11·9	11·7	11·3	11·0	10·9	10·7
0·004	9·89	9·80	9·60	9·41	9·20	8·90	8·73	8·60	8·45
0·007	8·25	8·17	8·00	7·82	7·64	7·40	7·29	7·16	7·04
0·01	7·39	7·31	7·14	6·99	6·81	6·60	6·46	6·38	6·28
0·02	6·04	5·97	5·81	5·67	5·54	5·39	5·29	5·22	5·14
0·04	5·09	5·02	4·88	4·77	4·66	4·54	4·47	4·41	4·35
0·07	4·59	4·53	4·41	4·31	4·23	4·13	4·07	4·02	3·97
0·1	4·41	4·34	4·23	4·15	4·07	3·99	3·94	3·90	3·86
0·2	4·27	4·22	4·14	4·07	4·01	3·94	3·90	3·86	3·81

$$\alpha = \frac{\epsilon\sigma T_0^3 R}{k}$$

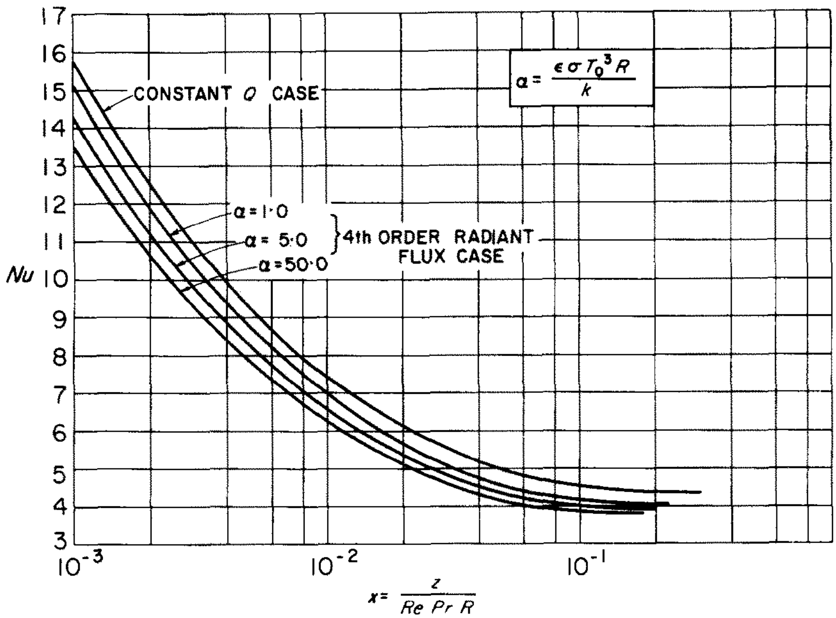

FIG. 2. Local Nusselt numbers.

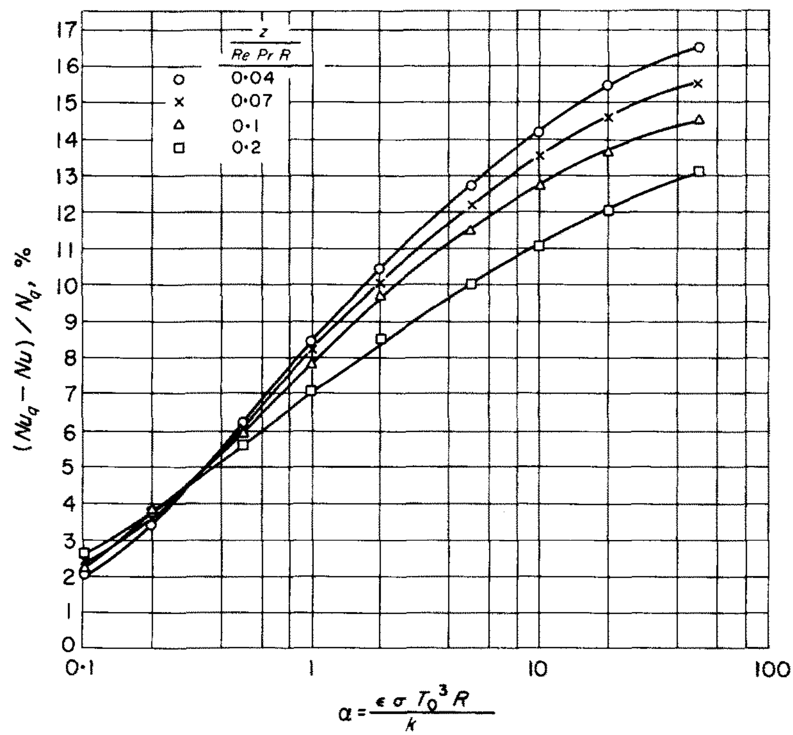

FIG. 3. Effect of α on decrease in Nu.

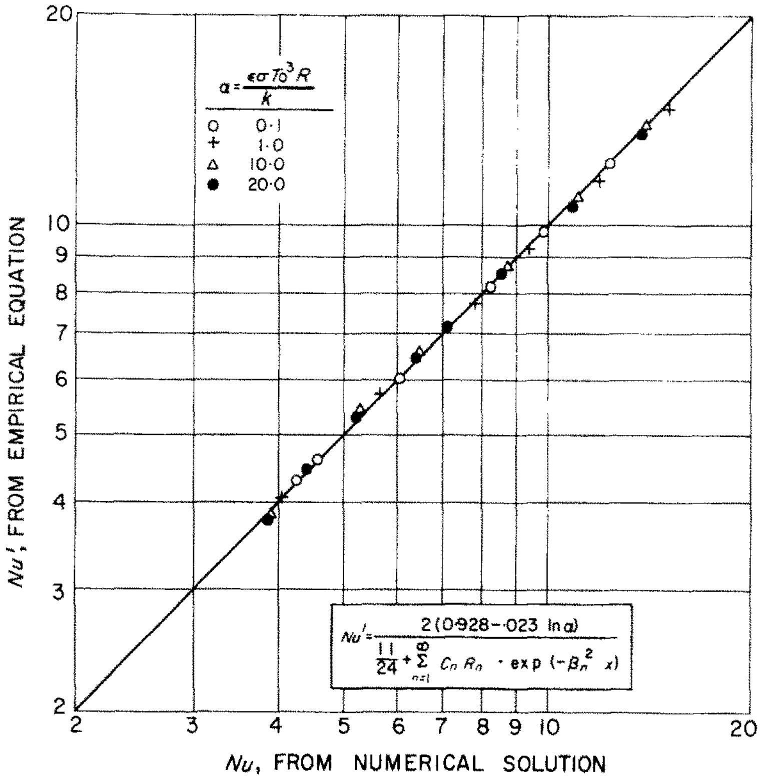

FIG. 4. Comparison of the empirical equation with numerical solution.

Local Nusselt numbers were calculated by equation (10) and are tabulated in Table 1. Figure 2 shows curves of the entrance region Nusselt numbers for three values of α. The curve for the constant heat flux case was recalculated, using 50 eigenconstants, and is shown here for comparison. It is evident that the radiant heat flux case has lower Nusselt numbers than the constant heat flux case. The difference is shown in Fig. 3 where the per cent decrease, relative to the constant heat flux case, is plotted as a function of the parameter α. For $0{\cdot}1 \leqslant \alpha \leqslant 50$, the difference ranges from 2 per cent to 17 per cent.

Inasmuch as these results are universal with just one independent variable, $x = (Z/R\ Re\ Pr)$, and one parameter, $\alpha = (\epsilon\sigma T_0^3 R/k)$, Table 1 and Fig. 2 may be used with interpolation for any case of interest. Figure 5 compares some approximate results calculated from equation (13) with the exact numerical results. The wall temperature is plotted as a function of axial position for two values of α (0·2 and 0·5). Two terms ($i = 0, 1$) were used in the Liouville–Neumann series for this calculation. For the range of variables shown, maximum deviation is 2 per cent.

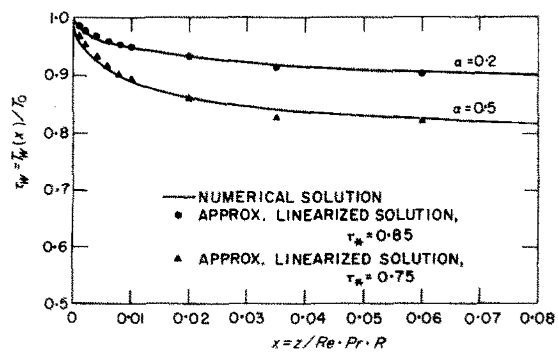

FIG. 5. Comparison of approximate and numerical solutions.

EMPIRICAL CORRELATION

It was found that the results of the numerical solution could be well correlated by the following equation

$$Nu(\alpha, x) = (a - b \ln \alpha)\, Nu_q(x) \qquad (17)$$

where $Nu_q(x)$ = Nusselt number for constant heat flux case

$$= \frac{2}{\frac{11}{24} + \sum_{n=1}^{\infty} C_n R_n \exp(-\beta_n^2 x)}$$

The constants were empirically determined to be

$$a = 0{\cdot}928; \quad b = 0{\cdot}023.$$

Figure 4 compares Nusselt numbers calculated by this equation with the Nusselt numbers obtained from the numerical solution. In the range of $0{\cdot}002 \leqslant x \leqslant 0{\cdot}2$ and for $\alpha \leqslant 20$, agreement is better than ± 2 per cent. This equation is convenient to use and sufficiently accurate for most applications.

SUMMARY

The problem of laminar heat transfer in a tube with the nonlinear radiant-flux boundary condition was analysed and an approximate solution presented in terms of the Liouville–Neumann series. Exact numerical solutions were also obtained for a wide range of the parameter, $\alpha = (\epsilon\sigma T_0^3 R/k)$. Results for the local Nusselt number are presented in tabular and graphical forms. An empirical equation is also given which permits rapid estimation of the local Nusselt number within an accuracy of 2 per cent for any $\alpha \leqslant 20$.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Professor R. D. Cess of State University of New York at Stony Brook and to Drs. C. J. Hsu and O. E. Dwyer of Brookhaven National Laboratory for helpful discussions during the course of this work.

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Résumé—Le problème du refroidissement par rayonnement d'un fluide en écoulement laminaire dans un tube a été décrit par une équation intégrale non linéaire, et une solution approchée a été obtenue sous la forme d'une série de Liouville–Neumann. On a obtenu aussi des résultats par une solution numérique exacte par itération. Les nombres de Nusselt locaux sont présentés comme fonctions de la distance sans dimensions, $x = Z/Re\ Pr\ R$, et d'un paramètre sans dimensions, $\alpha = \epsilon\sigma\ T_0^3/k$. Une équation empirique:

$$Nu(\alpha, x) = (0{,}928 - 0{,}023 \ln \alpha(Nu_q(x)$$

où Nu_q = nombre de Nusselt pour le cas du flux de chaleur constant, donne des résultats avec une précision de ± 2 pour cent dans les gammes de variables intéressantes.

Zusammenfassung—Es wird das Problem der Kühlung einer laminar durch ein Rohr strömenden Flüssigkeit durch Temperaturstrahlung beschrieben mit Hilfe einer nichtlinearen Integralgleichung und einer Näherungslösung in Form einer Liouville–Neumann Reihe. Ergebnisse wurden auch durch eine exakte iterative numerische Lösung erhalten. Örtliche Nusseltzahlen sind als Funktion des dimensionslosen Abstands $x = Z/Re\ Pr\ R$ und eines dimensionslosen Parameters $\alpha = \epsilon\sigma\ T_0^3/k$ angegeben. Eine empirische Gleichung

$$Nu(\alpha, x) = (0{,}928 - 0{,}023 \ln \alpha)\, Nu_q(x)$$

mit Nu_q = Nusseltzahl für konstanten Wärmefluss, liefert Ergebnisse die innerhalb ± 2 Prozent im Bereich der interessierenden Variablen genau sind.

Аннотация—Задача лучистого охлаждения жидкости при ламинарном течении в трубе описывается нелинейным интегральным уравнением. Получено приближенное решение в виде рядов Лиувилля–Неймана, а также с помощью точного итерационного численного решения. Локальные значения критерия Нуссельта представлены как функция безразмерного расстояния, $x = Z/RePrR$, и безразмерного параметра $a = \epsilon\sigma T_0^3/k$.

$$Nu(a,x) = (0{,}928 - 0{,}023\ ln\ a)\ Nu_q(x),$$

Найдено, что эмпирическое уравнение
где $Nu_q =$ критерий Нуссельта в случае постоянного теплового потока, дает результаты с точностью до $\pm 2\%$ для требуемых диапазонов переменных.